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# ON GRADED QUOTIENT MODULES OF MAPPING CLASS GROUPS OF SURFACES

BY

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ABSTRACT

Let $\Gamma_{g,n}$ be the mapping class group of a compact Riemann surface of genus $g$ with $n$ points preserved $(2-2g-n<0, g\geq 1, n\geq 0)$. The Torelli subgroup of $\Gamma_{g,n}$ has a natural weight filtration $\{\Gamma_{g,n}(m)\}_{m\geq 1}$. Each graded quotient $\mathrm{gr}^m\,\Gamma_{g,n}\otimes\mathbb{Q}$ $(m\geq 1)$ is a finite dimensional vector space over $\mathbb{Q}$ on which the group $\mathrm{Sp}(2g,\mathbb{Q})\times S_n$ naturally acts.

In this paper, we have determined the $\mathrm{Sp}(2g,\mathbb{Q})\times S_n$ module structure of $\mathrm{gr}^m\,\Gamma_{g,n}\otimes\mathbb{Q}$ for $1\leq m\leq 3$. This includes a verification of an expectation by S. Morita. Also, for general $m$, we have identified a certain $\mathrm{Sp}(2g,\mathbb{Q})$-irreducible component of $\mathrm{gr}^m\,\Gamma_{g,n}\otimes\mathbb{Q}$ by constructing explicitly elements in these modules.

## 1. Introduction

The purpose of this paper is to present some results on graded quotient modules of the weight filtrations in the mapping class groups of surfaces. Let $\Gamma_{g,n}$ be the mapping class group of a compact Riemann surface $R$ of genus $g$ with $n$ points

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preserved $(2-2g-n<0, g\geq 1, n\geq 0)$. The Torelli subgroup $\Gamma_{g,n}(1)$ has a weight filtration

$$\Gamma_{g,n}(1)\supset\Gamma_{g,n}(2)\supset\cdots$$

induced from the weight filtration of $\pi_1(R-\{n\text{-points}\})$. Each graded quotient $\mathrm{gr}^m_{\mathbb{Q}}\Gamma_{g,n}=\Gamma_{g,n}(m)/\Gamma_{g,n}(m+1)\otimes\mathbb{Q}$ $(m\geq 1)$ has a natural structure of a finite dimensional vector space and the associated graded sum $\mathrm{Gr}_{\mathbb{Q}}\Gamma_{g,n}=\bigoplus_{m=1}^{\infty}\mathrm{gr}^m_{\mathbb{Q}}\Gamma_{g,n}$ has a natural structure of a graded Lie algebra. Moreover, the group $\mathrm{Sp}(2g,\mathbb{Q})\times S_n$ naturally operates on both of them (cf. Lemma (2.2.8)). Here Sp denotes the symplectic group, and $S_n$ denotes the symmetric group of degree $n$. We denote by $[\lambda]_{\mathrm{Sp}(2g)}$ the rational irreducible representation of $\mathrm{Sp}(2g,\mathbb{Q})$ with highest weight corresponding to a partition $\lambda$, and by $\left[{n \atop k}\right]$ the representation of $S_n$ induced from the trivial representation of $S_k\times S_{n-k}$. (If $n<k$, we set $\left[{n \atop k}\right]=0$.) We will understand that these notations $[\lambda]_{\mathrm{Sp}(2g)},\left[{n \atop k}\right]$ also denote the representation spaces of $\mathrm{Sp}(2g,\mathbb{Q})\times S_n$ corresponding to them in the obvious manner. Our first result is the following

THEOREM A: *Assume $g\geq 3$. Then the following isomorphisms of $\mathrm{Sp}(2g,\mathbb{Q})\times S_n$-modules hold:*

(1) $\mathrm{gr}^1_{\mathbb{Q}}\Gamma_{g,n}=[1^3]_{\mathrm{Sp}(2g)}+\left[{n \atop 1}\right]\otimes[1]_{\mathrm{Sp}(2g)}$,
(2) $\mathrm{gr}^2_{\mathbb{Q}}\Gamma_{g,n}=[2^2]_{\mathrm{Sp}(2g)}+\left[{n \atop 1}\right]\otimes[1^2]_{\mathrm{Sp}(2g)}+\left[{n \atop 2}\right]\otimes[0]_{\mathrm{Sp}(2g)}$,
(3) $\mathrm{gr}^3_{\mathbb{Q}}\Gamma_{g,n}=[3,1^2]_{\mathrm{Sp}(2g)}+\left[{n \atop 1}\right]\otimes[2,1]_{\mathrm{Sp}(2g)}+\wedge^2\left[{n \atop 1}\right]\otimes[1]_{\mathrm{Sp}(2g)}$.

For the module $\mathrm{gr}^m_{\mathbb{Q}}\Gamma_{g,n}$ for general $m$, we have the following

THEOREM B: *Assume $g\geq 3$. Then there exist explicit elements $\bar{\alpha}\in\mathrm{gr}^1_{\mathbb{Q}}\Gamma_{g,n}$, $\bar{\beta},\bar{\gamma}\in\mathrm{gr}^2_{\mathbb{Q}}\Gamma_{g,n}$ such that*

(1) *$\mathrm{ad}(\bar{\gamma})^m(\bar{\alpha})$ gives a nontrivial element of $\mathrm{gr}^{1+2m}_{\mathbb{Q}}\Gamma_{g,n}$ which generates an $\mathrm{Sp}(2g)$-irreducible component of type $[1+2m,1^2]$ $(m\geq 0)$;*
(2) *$\mathrm{ad}(\bar{\gamma})^m(\bar{\beta})$ gives a nontrivial element of $\mathrm{gr}^{2+2m}_{\mathbb{Q}}\Gamma_{g,n}$ which generates an $\mathrm{Sp}(2g)$-irreducible component of type $[2+2m,2]$ $(m\geq 0)$.*

*Moreover, these irreducible components appear with multiplicity one.*

We shall explain related works of Johnson and of Morita from the viewpoint of the above results. Let $\Gamma^*_{g,1}$ be the mapping class group of a Riemann surface of genus $g$ with one fixed boundary component. Then $\Gamma^*_{g,1}$ is a central extension of $\Gamma_{g,1}$ by $\mathbb{Z}$. There is an induced filtration on $\Gamma^*_{g,1}$ such that $\mathrm{gr}^m\Gamma^*_{g,1}\cong\mathrm{gr}^m\Gamma_{g,1}$ except for $m=2$. In general, we have an embedding of $\mathrm{gr}^m\Gamma^*_{g,1}$ into $\mathrm{gr}^m\Gamma^{\mathrm{alg}*}_{g,1}$

which is an explicit $\mathrm{Sp}(2g,\mathbb{Q})$-module fitting into the exact sequence:

$$0 \to \mathrm{gr}^m \Gamma_{g,1}^{\mathrm{alg}*} \to \mathrm{Hom}(\mathrm{gr}^1_{\mathbb{Q}} \Pi_{g,1}, \mathrm{gr}^{m+1}_{\mathbb{Q}} \Pi_{g,1}) \to \mathrm{gr}^{m+2}_{\mathbb{Q}} \Pi_{g,1} \to 0.$$

In this terminology, part of the results of Johnson [J] and Morita [Mo1],[Mo2] can be summarized as follows. Firstly, Johnson showed that $\mathrm{gr}^1_{\mathbb{Q}} \Gamma_{g,1} \cong \mathrm{gr}^1 \Gamma_{g,1}^{\mathrm{alg}*}$ and then Morita proved that $\mathrm{gr}^2_{\mathbb{Q}} \Gamma^*_{g,1} \cong \mathrm{gr}^2 \Gamma_{g,1}^{\mathrm{alg}*}$. (Actually their results imply more about lattices.) In a more recent paper [Mo2], Morita also proved that, for odd $m \geq 3$, $\mathrm{gr}^m_{\mathbb{Q}} \Gamma^*_{g,1}$ is strictly smaller than $\mathrm{gr}^m \Gamma_{g,1}^{\mathrm{alg}*}$ by using his theory of trace maps. Our proof of Theorem A will be reduced to these results of Johnson and of Morita. Theorem A (3) verifies Morita's expectation which appeared in [Mo2] Remark 6.12 (i).

Recently, T. Oda [O] has shown, using the notion of 'cycle twists' due to M. Matsumoto, that $\mathrm{gr}^m \Gamma_{g,0}$ contains a submodule isomorphic to a graded quotient module of Artin's pure braid group with respect to the lower central series. It seems an interesting task to relate our Lie technique with their works to obtain further detailed description of the image of $\mathrm{gr}^m_{\mathbb{Q}} \Gamma_{g,0}$ in $\mathrm{gr}^m \Gamma_{g,0}^{\mathrm{alg}}$.

The organization of the present paper is as follows. Section 2 is devoted to relatively general arguments on the weight filtration in the mapping class group. In 2.1, we define the weight filtration and review its basic properties. In 2.2, the 'coordinate modules' are introduced, into which the graded quotient modules by the weight filtration are embedded. In 2.3, we explain how to calculate the $\mathrm{Sp}(2g) \times S_n$-representations in the algebraic graded quotients defined in the coordinate modules. In 2.4, we review Fox's free differential calculus, and then give a short introduction to Morita's theory of trace maps.

Section 3 deals with construction of special elements $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ of Theorem B. We first define two preliminary elements by using certain explicit Dehn twists, and then calculate their suitable deformations to obtain $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ under the actions of the symplectic Lie algebra through coodinate modules. Next, we examine that iterated Lie brackets of Theorem B give nontrivial vectors of certain good weights which will turn out to be highest.

Finally in Section 4, we synthesize the arguments in Sections 2 and 3 and conclude the proofs of Theorems A, B.

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*Complementary note:* In a recent conference "Topology of Moduli Spaces of Curves" at Kyoto in September 1993, we have learned that Prof. S. Morita has determined the $\mathrm{Sp}(2g,\mathbb{Q})$-module structure of $\mathrm{gr}^4_{\mathbb{Q}}\Gamma_{g,1}$.

## 2. Weight filtrations and Morita's trace maps

2.1. BASIC SETUP.

(2.1.1). Let $\Pi_{g,n}$ be the fundamental group of an $n$-point punctured Riemann surface of genus $g$ presented by

$$\Pi_{g,n}=\left\langle \begin{matrix} x_1,\dots,x_g,x_{g+1},\dots,x_{2g} \\ z_1,\dots,z_n \end{matrix} \,\middle|\, [x_1,x_{g+1}]\cdots[x_g,x_{2g}]z_1\cdots z_n=1\right\rangle .$$

Here $[,]$ denotes the commutator bracket: $[x,y]=xyx^{-1}y^{-1}$. We assume $g\geq 1$, $n\geq 0$ and $2-2g-n<0$, and introduce a central filtration (due to Oda–Kaneko [K]) $\Pi_{g,n}=\Pi_{g,n}(1)\supset\Pi_{g,n}(2)\supset\cdots$, called **weight filtration**, by setting

$$\begin{aligned} \Pi_{g,n}(1)&=\Pi_{g,n},\\ \Pi_{g,n}(2)&=[\Pi_{g,n},\Pi_{g,n}]\langle z_1,\dots,z_n\rangle,\\ \Pi_{g,n}(m)&=\langle [g_1,g_2] \mid g_1\in\Pi_{g,n}(m'),g_2\in\Pi_{g,n}(m''),(m'+m''=m)\rangle \quad (m\geq 3). \end{aligned}$$

It is known that $\bigcap_{m\geq 1}\Pi_{g,n}(m)=\{1\}$. Notice that when $n=0,1$ this central filtration coincides with the usual lower central filtration.

(2.1.2) Let $\mathrm{gr}^m\Pi_{g,n}=\Pi_{g,n}(m)/\Pi_{g,n}(m+1)$ and $\mathrm{Gr}\,\Pi_{g,n}=\bigoplus_{m=1}^{\infty}\mathrm{gr}^m\Pi_{g,n}$. Then $\mathrm{Gr}\,\Pi_{g,n}$ has a structure of a graded Lie algebra over $\mathbb{Z}$ induced from the commutator bracket on $\Pi_{g,n}$. It follows from Labute [L] that $\mathrm{Gr}\,\Pi_{g,n}$ is presented by

$$\begin{aligned} \text{generators: } & X_i=x_i \bmod \Pi_{g,n}(2)\quad (i=1,\cdots,2g),\\ & Z_j=z_j \bmod \Pi_{g,n}(3)\quad (j=1,\cdots,n);\\ \text{relation: } & \sum_{i=1}^{g}[X_i,X_{g+i}]+\sum_{j=1}^{n}Z_j=0, \end{aligned}$$

and that for each $m$, $\operatorname{gr}^m \Pi_{g,n}$ is a finitely generated free $\mathbb{Z}$-module whose rank is given by

$$\text{(2.1.3)} \quad \operatorname{rank} \operatorname{gr}^m \Pi_{g,n} = \frac{1}{m} \sum_{d|m} \mu\Big(\frac{m}{d}\Big) \sum_{0 \le i \le [\frac{d}{2}]} \frac{d}{d-i} \binom{d-i}{i} (n-1)^i \, (2g)^{d-2i}$$

(cf. [K]).

(2.1.4) Define the group of "braid-like" automorphisms of $\Pi_{g,n}$ by

$$\tilde{\Gamma}_{g,[n]} = \{\sigma \in \operatorname{Aut} \Pi_{g,n} \big| \, \sigma(z_j) \sim z^{\alpha}_{\pi(j)} (j = 1, \ldots, n) \; \exists \, \alpha = \pm 1, \, \exists \, \pi = \pi_\sigma \in S_n\},$$

where $\sim$ denotes conjugacy in $\Pi_{g,n}$. We consider that $\tilde{\Gamma}_{g,[n]}$ acts on $\Pi_{g,n}$ on the left. Since each element of $\tilde{\Gamma}_{g,[n]}$ stabilizes $\Pi_{g,n}(2)$, $\tilde{\Gamma}_{g,[n]}$ acts on $\Pi_{g,n}/\Pi_{g,n}(2) \simeq \mathbb{Z}^{2g}$. From this we get a representation

$$\begin{aligned} \rho \colon \tilde{\Gamma}_{g,[n]} &\longrightarrow \mathrm{GL}(2g, \mathbb{Z}) \\ \sigma &\longmapsto \rho_\sigma = (\rho_{ij}) \end{aligned}$$

by $\sigma(x_i) \equiv \prod_{k=1}^{2g} x_k^{\rho_{ki}} \mod \Pi_{g,n}(2)$. It follows by a result of Nielsen that the image of this representation coincides with

$$\mathrm{GSp}(2g, \mathbb{Z}) = \left\{ A \in \mathrm{GL}(2g, \mathbb{Z}) \,\middle|\, {}^t A J_g A = \chi(A) J_g, \, J_g = \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \right\},$$

where $\chi(A) = \pm 1$.

(2.1.5) Now we introduce groups $\tilde{\Gamma}_{g,n}$, $\Gamma_{g,n}$, $\Gamma^*_{g,n}$ defined by

$$\begin{aligned} \tilde{\Gamma}_{g,n} &= \{\sigma \in \tilde{\Gamma}_{g,[n]} | \pi_\sigma = \mathrm{id}\}, \\ \Gamma_{g,n} &= \{\sigma \in \tilde{\Gamma}_{g,n} | \chi(\rho_\sigma) = 1, \, \pi_\sigma = \mathrm{id}\} / \operatorname{Int} \Pi_{g,n}, \\ \Gamma^*_{g,n} &= \{\sigma \in \tilde{\Gamma}_{g,n} | \chi(\rho_\sigma) = 1, \sigma(z_n) = z_n, \, \pi_\sigma = \mathrm{id}\}. \end{aligned}$$

It is well-known that $\Gamma_{g,n}$ (resp. $\Gamma^*_{g,n}$) is the mapping class group of a genus $g$ surface with $n$-points preserved (resp. with $(n-1)$-points preseved and 1 boundary component fixed) (cf. Birman [Bi]).

(2.1.6) We give $\tilde{\Gamma}_{g,[n]}$ a filtration

$$\tilde{\Gamma}_{g,[n]} \supset \tilde{\Gamma}_{g,n}(1) \supset \tilde{\Gamma}_{g,n}(2) \supset \cdots$$

by setting

$$\tilde{\Gamma}_{g,n}(m) = \left\{ \sigma \in \tilde{\Gamma}_{g,n} \,\middle|\, \begin{array}{c} \sigma(x_i)x_i^{-1} \in \Pi_{g,n}(m+1)(i=1,\ldots,2g) \\ \sigma(z_j) \overset{m}{\sim} z_j (j=1,\ldots,n) \end{array} \right\} \quad (m \geq 1),$$

where $\overset{m}{\sim}$ denotes conjugacy by an element in $\Pi_{g,n}(m)$. Then, on $\Gamma_{g,n}$ and $\Gamma^*_{g,n}$, natural filtrations are induced from that on $\tilde{\Gamma}_{g,n}$.

(2.1.7) For each $m$, $\mathrm{gr}^m \Pi_{g,n}$ is acted on by, in general, $\mathrm{GSp}(2g,\mathbb{Z}) \times S_n$ naturally in the following manner. Since the filtration $\{\Pi_{g,n}(m)\}$ is preserved by the action of $\tilde{\Gamma}_{g,[n]}$, each $\mathrm{gr}^m \Pi_{g,n}$ can be a $\tilde{\Gamma}_{g,[n]}$-module. But it follows from the triviality of the action of $\tilde{\Gamma}_{g,n}(1)$ on $\Pi_{g,n}/\Pi_{g,n}(2)$ that the above operation of $\tilde{\Gamma}_{g,[n]}$ on $\mathrm{gr}^m \Pi_{g,n}$ factors through $\tilde{\Gamma}_{g,[n]}/\tilde{\Gamma}_{g,n}(1) \cong \mathrm{Sp}(2g,\mathbb{Z}) \times S_n$. Observe that only when $n=1$, $\mathrm{gr}^m \Pi_{g,1}$ can be a natural $\mathrm{GL}(2g,\mathbb{Z})$-module.

(2.1.8) We can also consider $\mathrm{gr}^m \tilde{\Gamma}_{g,n} = \tilde{\Gamma}_{g,n}(m)/\tilde{\Gamma}_{g,n}(m+1)$ $(m \geq 1)$ and $\mathrm{Gr}\,\tilde{\Gamma}_{g,n} = \bigoplus_{m=1}^{\infty} \mathrm{gr}^m \tilde{\Gamma}_{g,n}$. Since the weight filtration in $\tilde{\Gamma}_{g,n}(1)$ is central, $\mathrm{Gr}\,\tilde{\Gamma}_{g,n}$ has a structure of a graded Lie algebra. Each element $\sigma$ of $\tilde{\Gamma}_{g,n}(m)$ induces a derivation $D_\sigma$ of $\mathrm{Gr}\,\Pi_{g,n}$ by

$$D_\sigma(x \bmod \Pi_{g,n}(i+1)) = \sigma(x)x^{-1} \bmod \Pi_{g,n}(i+m+1) \quad (i \geq 1).$$

Then the mapping

$$D\colon \mathrm{Gr}\,\tilde{\Gamma}_{g,n} \to \mathrm{Der}(\mathrm{Gr}\,\Pi_{g,n})$$

gives an injective Lie algebra homomorphism. Here Der( ) denotes the Lie algebra of graded derivations of the Lie algebra inside. (cf. Bourbaki [Bo] Chap. 2, Ex §4-3, see also Nakamura–Tsunogai [NT1] §5.)

(2.1.9) Similarly, we introduce the notations $\mathrm{gr}^m \Gamma^*_{g,n}$, $\mathrm{Gr}\,\Gamma^*_{g,n}$, $\mathrm{gr}^m \Gamma_{g,n}$ and $\mathrm{Gr}\,\Gamma_{g,n}$ by which we mean obvious senses. Notice that $\mathrm{Gr}\,\Gamma^*_{g,n}$ is a Lie subalgebra of $\mathrm{Gr}\,\tilde{\Gamma}_{g,n}$ but $\mathrm{Gr}\,\Gamma_{g,n}$ is its quotient by $\mathrm{Gr}\,\Pi_{g,n}$ (cf. [NT1]). Observe also that by conjugation the graded quotients $\mathrm{gr}^m \tilde{\Gamma}_{g,n}$, $\mathrm{gr}^m \Gamma^*_{g,n}$ and $\mathrm{gr}^m \Gamma_{g,n}$ are naturally $\mathrm{GSp}(2g,\mathbb{Z}) \times S_n$-modules. (Unlike $\mathrm{gr}^m \Pi_{g,1}$, the modules $\mathrm{gr}^m \tilde{\Gamma}_{g,1}$ etc. are not $\mathrm{GL}(2g,\mathbb{Z})$-modules apriori.) As $\mathbb{Z}$-modules, $\mathrm{gr}^m \Gamma_{g,n}$, $\mathrm{gr}^m \tilde{\Gamma}_{g,n}$ (hence $\mathrm{gr}^m \Gamma^*_{g,n}$) are free of finite rank (Asada [A2]). We see that $\Gamma^*_{g,n}$ is a central extension of $\Gamma_{g,n}$ by $\mathbb{Z}$, and that $\mathrm{gr}^m \Gamma_{g,n} \cong \mathrm{gr}^m \Gamma^*_{g,n}$ when $m \neq 2$.

2.2. Coordinate modules and fundamental complexes.

(2.2.1) To describe the module $\mathrm{gr}^m \Gamma_{g,n}$ $(m \geq 1)$, let us recall basic facts about the coordinate module and its submodules (cf. [NT1]). The **coordinate module** $C^m(2g,n)$ $(m \geq 1)$ is defined by

$$C^m(2g,n) = \begin{cases} (\mathrm{gr}^{m+1}\Pi_{g,n})^{\oplus 2g} \oplus (\mathrm{gr}^m \Pi_{g,n})^{\oplus n} & (m \neq 2), \\ (\mathrm{gr}^{m+1}\Pi_{g,n})^{\oplus 2g} \oplus \bigoplus_{j=1}^{n} (\mathrm{gr}^m \Pi_{g,n}/\langle \bar{z}_j \rangle) & (m = 2), \end{cases}$$

and its $\mathrm{Sp}(2g,\mathbb{Z}) \times S_n$-module strucutre is introduced by

$$(2.2.2) \qquad (\rho,\pi)\cdot(S_i)_i \times (T_j)_j := \Big(\sum_{k=1}^{2g} \rho_{ki}(\rho,\pi)(S_k)\Big)_i \times ((\rho,\pi)(T_{\pi^{-1}(j)}))_j$$

where $\rho = (\rho_{ij}) \in \mathrm{Sp}(2g,\mathbb{Z})$, $\pi \in S_n$, and $(S_i)_i \times (T_j)_j = (S_1,\ldots,S_{2g},T_1,\ldots,T_n)$ is an element of $C^m(2g,n)$.

(2.2.3) Let $\tilde{f}_m$ denote the following homomorphism

$$\tilde{f}_m \colon C^m(2g,n) \longrightarrow \mathrm{gr}^{m+2}\Pi_{g,n}$$
$$(S_i)_i \times (T_j)_j \longmapsto \sum_{i=1}^{g}([X_i,S_{g+i}] + [S_i,X_{g+i}]) + \sum_{j=1}^{n}[T_j,Z_j],$$

and denote the kernel of $\tilde{f}_m \otimes \mathbb{Q}$ by $\mathrm{gr}^m \tilde{\Gamma}^{\mathrm{alg}}_{g,n}$. (Note that we do not define $\tilde{\Gamma}^{\mathrm{alg}}_{g,n}$ itself in this paper.) We define the mapping

$$c_m \colon \tilde{\Gamma}_{g,n}(m) \to C^m(2g,n) \qquad (m \geq 1)$$

as follows. For $\sigma \in \tilde{\Gamma}_{g,n}(m)$, put $s_i(\sigma) = \sigma(x_i)x_i^{-1}$ $(1 \leq i \leq 2g)$, and let $t_j(\sigma)$ be an element of $\Pi_{g,n}(m)$ such that $\sigma(z_j) = t_j(\sigma)z_jt_j(\sigma)^{-1}$ $(1 \leq j \leq n)$. Since the centralizer of $z_j$ in $\Pi_{g,n}$ is $\langle z_j \rangle$, $t_j(\sigma) \bmod \Pi_{g,n}(m+1)$ is uniquely determined if $m \neq 2$ and $t_j(\sigma) \bmod \Pi_{g,n}(3)\langle z_j \rangle$ is uniquely determined if $m = 2$. Then, letting $\bar{}$ represent suitable quotient images, we define

$$c_m(\sigma) = (\bar{s}_i(\sigma))_{1 \leq i \leq 2g} \times (\bar{t}_j(\sigma))_{1 \leq j \leq n}.$$

Since $s_i(\sigma\tau) = \sigma(s_i(\tau))s_i(\sigma)$, $t_j(\sigma\tau) = \sigma(t_j(\tau))t_j(\sigma)$ $(\sigma,\tau \in \tilde{\Gamma}_{g,n}(m))$ and since $\tilde{\Gamma}_{g,n}(m)$ acts trivially on $\mathrm{Gr}\,\Pi_{g,n}$, the map $c_m$ induces an injective homomorphism

$$\tilde{\iota}_m \colon \mathrm{gr}^m \tilde{\Gamma}_{g,n} \to C^m(2g,n).$$

We note that, by definition, $\bar{s}_i(\sigma)$ is nothing but $D_\sigma(x_i)$, $D_\sigma$ being the derivation associated with $\sigma$ (cf. (2.1.8)).

The following lemma is basic and can be found in e.g. [K] (in the context of pro-$l$ groups), [Mo2] Corollary 3.2 (in the case of $n = 1$).

LEMMA (2.2.4): *The image of* $\mathrm{gr}^m \tilde{\Gamma}_{g,n}$ *via* $\tilde{\iota}_m$ *is contained in* $\mathrm{Ker}\, \tilde{f}_m$.

PROPOSITION (2.2.5): *The above homomorphisms* $\tilde{\iota}_m$ *and* $\tilde{f}_m$ *are equivariant with the actions of* $\mathrm{Sp}(2g, \mathbb{Z}) \times S_n$. *Hence we have a complex of* $\mathrm{Sp}(2g, \mathbb{Z}) \times S_n$*-modules:*

$$\mathrm{gr}^m \tilde{\Gamma}_{g,n} \xrightarrow{\tilde{\iota}_m} C^m(2g, n) \xrightarrow{\tilde{f}_m} \mathrm{gr}^{m+2} \Pi_{g,n}$$

*in which* $\tilde{\iota}_m$ *is injective and* $\tilde{f}_m$ *is surjective.*

*Proof:* The injectivity of $\tilde{\iota}_m$ is obvious and the inclusion $\mathrm{Im}\, \tilde{\iota}_m \subset \mathrm{Ker}\, \tilde{f}_m$ follows from Lemma (2.2.4). The surjectivity of $\tilde{f}_m$ follows from the fact that the Lie algebra $\mathrm{Gr}\, \Pi_{g,n}$ is generated by $X_i$ $(1 \le i \le 2g)$, $Z_j$ $(1 \le j \le n-1)$. For the $\mathrm{Sp}(2g) \times S_n$-equivariances of $\tilde{\iota}_m$ and $\tilde{f}_m$, see [NT1] §1, Theorem (1.14). ∎

Now, we define

$$C^m(2g, n)^* = \{(S_i) \times (T_j) \in C^m(2g, n) | T_n = 0\},$$

which is a submodule of $C^m(2g, n)$. Then,

COROLLARY (2.2.6): *For* $m \ge 1$, *the sequence*

$$\mathrm{gr}^m \Gamma^*_{g,n} \xrightarrow{\iota^*_m} C^m(2g, n)^* \xrightarrow{f^*_m} \mathrm{gr}^{m+2} \Pi_{g,n}$$

*is a complex, where* $\iota^*_m$ *(resp.* $f^*_m$*) is the restriction of* $\tilde{\iota}_m$ *(resp.* $\tilde{f}_m$*). Moreover,* $\iota^*_m$ *is injective and* $f^*_m$ *is surjective, and both of them are* $\mathrm{Sp}(2g, \mathbb{Z})$*-equivariant.*

Since the Lie algebra $\mathrm{Gr}\, \Pi_{g,n}$ has trivial center (cf. [A2]), we can regard $\mathrm{gr}^m \Pi_{g,n}$ as a submodule of the coordinate module $C^m(2g, n)$ given by

$$\{([h, \bar{x}_i])_i \times (h, \ldots, h) \in C^m(2g, n) |\, h \in \mathrm{gr}^m \Pi_{g,n}\}.$$

The quotient $C^m(2g, n)/ \mathrm{gr}^m \Pi_{g,n}$ is called the **reduced coordinate module**, and the maps $\tilde{\iota}_m$, $\tilde{f}_m$ also induce the complex of $\mathrm{Sp}(2g, \mathbb{Z}) \times S_n$-modules:

$$\text{(2.2.7)} \qquad \mathrm{gr}^m \Gamma_{g,n} \xrightarrow{\iota_m} C^m(2g, n)/ \mathrm{gr}^m \Pi_{g,n} \xrightarrow{f_m} \mathrm{gr}^{m+2} \Pi_{g,n},$$

in which $\iota_m$ is injective and $f_m$ is surjective. We define $\mathrm{gr}^m \Gamma_{g,n}^{\mathrm{alg}}$ to be the kernel of $f_m \otimes \mathbb{Q}$. We also define $\mathrm{gr}^m \Gamma_{g,n}^{\mathrm{alg}*}$ to be the kernel of $f_m^* \otimes \mathbb{Q}$ in (2.2.6). It is not difficult to see that the following isomorphisms hold:

$$\mathrm{gr}^m \Gamma_{g,n}^{\mathrm{alg}*} \cong \mathrm{gr}^m \Gamma_{g,n}^{\mathrm{alg}} (m \neq 2), \quad \mathrm{gr}^2 \Gamma_{g,n}^{\mathrm{alg}*}/\langle \bar{z}_n \rangle \cong \mathrm{gr}^2 \Gamma_{g,n}^{\mathrm{alg}}.$$

By construction, we easily see that the actions of $\mathrm{Sp}(2g, \mathbb{Z})$ on the various coordinate modules are rational and that $\mathrm{gr}^m \tilde{\Gamma}_{g,n}^{\mathrm{alg}}$, $\mathrm{gr}^m \Gamma_{g,n}^{\mathrm{alg}*}$, $\mathrm{gr}^m \Gamma_{g,n}^{\mathrm{alg}}$ are rational representations of $\mathrm{Sp}(2g, \mathbb{Q})$ so that $\tilde{f}_m \otimes \mathbb{Q}$, $f_m^* \otimes \mathbb{Q}$, $f_m \otimes \mathbb{Q}$ are $\mathrm{Sp}(2g, \mathbb{Q})$-equivariant homomorphisms. On the other hand, it is a nontrivial problem whether the natural actions of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathrm{gr}^m \tilde{\Gamma}_{g,n}$, $\mathrm{gr}^m \Gamma_{g,n}^*$, $\mathrm{gr}^m \Gamma_{g,n}$ are rational ones or not. In this paper, we content ourselves with a weaker algebraicity result on those $\mathbb{Q}$-tensored modules which can be stated as follows.

ALGEBRAICITY LEMMA (2.2.8): *The images of $\tilde{\iota}_m \otimes \mathbb{Q}$, $\iota_m^* \otimes \mathbb{Q}$, $\iota_m \otimes \mathbb{Q}$ are stable under the natural actions of $\mathrm{Sp}(2g, \mathbb{Q})$ on the respective coordinate modules containing them. Hence, in particular, we may define (algebraic) actions of $\mathrm{Sp}(2g, \mathbb{Q})$ on $\mathrm{gr}_{\mathbb{Q}}^m \tilde{\Gamma}_{g,n}$, $\mathrm{gr}_{\mathbb{Q}}^m \Gamma_{g,n}^*$, $\mathrm{gr}_{\mathbb{Q}}^m \Gamma_{g,n}$ by restriction so that $\tilde{\iota}_m \otimes \mathbb{Q}$, $\iota_m^* \otimes \mathbb{Q}$, $\iota_m \otimes \mathbb{Q}$ are $\mathrm{Sp}(2g, \mathbb{Q})$-equivariant homomorphisms.*

*Proof:* We have only to see a general fact, that if $L$ is a $\mathbb{Z}$-submodule of a rational finite dimensional $\mathrm{Sp}(2g, \mathbb{Q})$-module $V$ which is stable under the action of $\mathrm{Sp}(2g, \mathbb{Z})$, then $L \otimes \mathbb{Q} \subset V$ is also stable under the action of $\mathrm{Sp}(2g, \mathbb{Q})$. In fact, suppose on the contrary that $Av \notin L \otimes \mathbb{Q}$ for some $v \in L \otimes \mathbb{Q}$ and $A \in \mathrm{Sp}(2g, \mathbb{Q})$. Then for a sufficiently large prime $p$ we have $A \in \mathrm{Sp}(2g, \mathbb{Z}_p)$. But *by the approximation theorem*, there is a sequence $\{A_i\}_{i=1}^{\infty} \subset \mathrm{Sp}(2g, \mathbb{Z})$ such that $\lim_{i \to \infty} A_i = A$ in the congruence topology of $\mathrm{Sp}(2g, \mathbb{Z}_p)$. Since $L \otimes \mathbb{Q}_p$ is closed in $V \otimes \mathbb{Q}_p$, we get $L \otimes \mathbb{Q}_p \ni \lim_{i \to \infty} A_i v = Av \notin L \otimes \mathbb{Q}_p$ which is a contradiction. ∎

*Remark* (2.2.9): The above maps $\tilde{\iota}_m$, $\iota_m^*$ and $\iota_m$ can be considered as generalizations of the Johnson homomorphisms

$$\tau_{m+1}\colon \mathrm{gr}^m \Gamma_{g,1}^* \to \mathrm{Hom}(\mathrm{gr}^1 \Pi_{g,1}, \mathrm{gr}^{m+1} \Pi_{g,1})$$

studied by Johnson [J] and Morita [Mo1]. ∎

### 2.3. Universal characters of weight graduations.

(2.3.1) Let us decompose the $\mathrm{Sp}(2g,\mathbb{C})\times S_n$-modules $\mathrm{gr}^m_{\mathbb{C}}\Pi_{g,n}=\mathrm{gr}^m\Pi_{g,n}\otimes\mathbb{C}$ and $\mathrm{gr}^m_{\mathbb{C}}\Gamma^{\mathrm{alg}}_{g,n}=\mathrm{gr}^m\Gamma^{\mathrm{alg}}_{g,n}\otimes\mathbb{C}$ as

$$\mathrm{gr}^m_{\mathbb{C}}\Pi_{g,n}=\sum_\lambda V_\lambda\otimes[\lambda]_{\mathrm{Sp}(2g)},$$
$$\mathrm{gr}^m_{\mathbb{C}}\Gamma^{\mathrm{alg}}_{g,n}=\sum_\lambda W_\lambda\otimes[\lambda]_{\mathrm{Sp}(2g)},$$

in which $\lambda$ run over all partitions, and $V_\lambda, W_\lambda$ are representation spaces of $S_n$ whose dimensions give multiplicities of $[\lambda]_{\mathrm{Sp}(2g)}$. (We regard each of these representations naturally as an $\mathrm{Sp}(2g)\times S_n$-representation.) For each partition $\pi$, we obtain the virtual characters of $\mathrm{Sp}(2g,\mathbb{C})$ given by

$$[\mathrm{gr}^m_{\mathbb{C}}\Pi_{g,n}(\pi)]:=\sum_\lambda \mathrm{Tr}(\pi_0,V_\lambda)\,[\lambda]_{\mathrm{Sp}(2g)},$$
$$[\mathrm{gr}^m_{\mathbb{C}}\Gamma^{\mathrm{alg}}_{g,n}(\pi)]:=\sum_\lambda \mathrm{Tr}(\pi_0,W_\lambda)\,[\lambda]_{\mathrm{Sp}(2g)},$$

where $\mathrm{Tr}(\pi_0,*)$ is the trace of the endomorphism on $*$ induced by $\pi_0\in S_n$ of cycle type $\pi$.

We wish to deal with these characters in a uniform way with respect to the genus $g$. In fact, as explained in the following, there exist universal characters $[\mathrm{gr}^m\Pi(\pi)]$, $[\mathrm{gr}^m\Gamma(\pi)]$ in the universal character ring for the classical Lie groups in the sense of Koike–Terada [KT] so that their 'specializations to $\mathrm{Sp}(2g)$' give the virtual characters $[\mathrm{gr}^m_{\mathbb{C}}\Pi_{g,n}(\pi)]$, $[\mathrm{gr}^m_{\mathbb{C}}\Gamma^{\mathrm{alg}}_{g,n}(\pi)]$.

(2.3.2) Let $\Lambda_n$ be the ring of symmetric functions in $n$ variables $t_1,\dots,t_n$ ($n\geq 1$), and put $\Lambda=\varprojlim\Lambda_n$ where the projective limit is taken with respect to the maps $\Lambda_n\to\Lambda_{n-1}$ which sends $t_n$ to 0 and keeps the $t_i$ ($1\le i\le n-1$) identically. This ring is introduced in Macdonald's book [Ma] and, as in the paper by Koike–Terada [KT], it can be considered as the universal character ring for classical Lie groups. For each partition $\pi$ and an integer $m\geq 1$, we can define an element $[\mathrm{gr}^m\Pi(\pi)]\in\Lambda$ by the formula

$$(2.3.3)\quad [\mathrm{gr}^m\Pi(\pi)]=\frac{1}{m}\sum_{d|m}\mu\Big(\frac{m}{d}\Big)\sum_{0\le i\le[d/2]}\frac{d}{d-i}\binom{d-i}{i}(F(\pi^{m/d})-1)^i\,p_{m/d}^{d-2i}$$

where $p_i$ $(i = 1, 2, \dots)$ is the power sum symmetric function of degree $i$, and $F(\pi^{m/d})$ is the sum of the parts of $\pi$ dividing $m/d$. We also define

(2.3.4)

$$[\mathrm{gr}^m \Gamma(\pi)] := p_1[\mathrm{gr}^{m+1} \Pi(\pi)] + (F(\pi) - 1)[\mathrm{gr}^m \Pi(\pi)] - [\mathrm{gr}^{m+2} \Pi(\pi)] - F(\pi)\delta_{2,m},$$

where $\delta$ means Kronecker's delta.

(2.3.5) Let $R(\mathrm{Sp}(2g))$ be the rational character ring of $\mathrm{Sp}(2g, \mathbb{C})$. In [KT], Koike and Terada introduced a specialization homomorphism

$$\pi_{\mathrm{Sp}(2g)}: \Lambda \to R(\mathrm{Sp}(2g)),$$

and described its precise behavior with the 'Young-diagramatic' method.

LEMMA (2.3.6): *Let $\pi$ be a partition with size $n = |\pi|$. Then,*

$$\pi_{\mathrm{Sp}(2g)}([\mathrm{gr}^m \Pi(\pi)]) = [\mathrm{gr}^m_{\mathbb{C}} \Pi_{g,n}(\pi)],$$
$$\pi_{\mathrm{Sp}(2g)}([\mathrm{gr}^m \Gamma(\pi)]) = [\mathrm{gr}^m_{\mathbb{C}} \Gamma^{\mathrm{alg}}_{g,n}(\pi)].$$

The first formula of this lemma can be obtained by modifying suitably the rank-formula (2.1.3) (cf. [KO] Lemma (6.3), [NT1] Remark (1.18)). The second one follows as a consequence of our definition of $\mathrm{gr}^m \Gamma^{\mathrm{alg}}_{g,n}$. A more leisured account of them will be presented in the forthcoming monograph [NT2]. From Lemma (2.3.6), we see that types of irreducible decompositions of $\mathrm{gr}^m_{\mathbb{C}} \Pi_{g,n}$, $\mathrm{gr}^m_{\mathbb{C}} \Gamma^{\mathrm{alg}}_{g,n}$ become stable when the genus $g$ increases sufficiently.

COROLLARY (2.3.7): *Let $g \geq 3$. An irreducible component with highest weight vectors of the $\mathrm{Sp}(2g, \mathbb{C})$-module $\mathrm{gr}^m_{\mathbb{C}} \Gamma^{\mathrm{alg}}_{g,n}$ appears with multiplicity one. It is $[m, 1^2]_{\mathrm{Sp}(2g)}$ when $m$ is odd and $[m, 2]_{\mathrm{Sp}(2g)}$ when $m$ is even.*

*Proof:* Observing the Laurent polynomial obtained by specializing $p_i$ to $(t_1^i + t_1^{-i} + \cdots + t_g^i + t_g^{-i})$ in the formula (2.3.4), we find that the coefficients of the monomials $t_1^{m+2}$, $t_1^m t_2^2$, $t_1^m t_2 t_3$ in it are respectively 0,0,1 when $m$ is odd and 0,1,1 when $m$ is even. These are multiplicities of the respective weights of the action of the standard torus of $\mathrm{Sp}(2g)$. The conclusion follows from this observation and the rule of specialization described in [KT] p.504. ∎

## 2.4. FREE DIFFERENTIAL CALCULUS, MORITA'S TRACE MAPS.

Let $R$ be the noncommutative formal power series ring over $\mathbb{Z}$ in variables $u_1, \dots, u_{2g}$. Then we can embed $\Pi_{g,1}$ as a subgroup of the unit group $R^\times$ of $R$

by sending $x_i$ to $1+u_i$ $(i=1,\dots,2g)$ (Magnus embedding). Let $\epsilon\colon R\to\mathbb{Z}$ be the augmentation map and $I=\ker(\epsilon)$ the augmentation ideal of $R$. For each $r\in R$, there exist elements $\partial r/\partial x_i\in R$ $(i=1,\dots,2g)$ uniquely determined by the formula

$$r=\epsilon(r)+\sum_i\frac{\partial r}{\partial x_i}(x_i-1).$$

(These $\partial r/\partial x_i$ are called Fox's free derivatives of $r$.)

By making use of free differential calculus, S.Morita [Mo2] introduced "trace" homomorphisms

$$\mathrm{Tr}_m: C^m(2g,1)^*\to \mathrm{Sym}^m H \quad (m\geq 1),$$

where $\mathrm{Sym}^m H$ denotes the $m$-th symmetric tensor of $H:=\Pi_{g,1}/\Pi_{g,1}(2)$. The definition of $\mathrm{Tr}_m$ is as follows. For each $S=(S_1,\dots,S_{2g},0)\in C^m(2g,1)^*$, choose lifts $s_i$ of $S_i$ in $\Pi_{g,1}(m+1)$ respectively. Then $\sum_i\partial s_i/\partial x_i$ gives an element of $I^m/I^{m+1}\cong H^{\otimes m}$ which is determined independently of the choice of the $s_i$. The trace $\mathrm{Tr}_m(S)$ is defined to be the image of $\sum_i\partial s_i/\partial x_i$ mod $I^{m+1}$ by the natural projection $H^{\otimes m}\to\mathrm{Sym}^m H$. In [Mo2], it is shown that $\mathrm{Tr}_m: C^m(2g,1)^*\to\mathrm{Sym}^m H$ is a $\mathrm{GL}(2g,\mathbb{Z})$-equivariant homomorphism satisfying the following properties.

(2.4.1) If $m$ is even, then $\mathrm{Tr}_m$ is 0-map on $\mathrm{gr}^m\,\Gamma_{g,1}^{\mathrm{alg}*}$.

(2.4.2) If $m$ is odd $\geq 1$, then $\mathrm{Tr}_m\otimes\mathbb{Q}$: $\mathrm{gr}^m\,\Gamma_{g,1}^{\mathrm{alg}*}\to\mathrm{Sym}^m H\otimes\mathbb{Q}$ is surjective.

(2.4.3) For $f\in C^n(2g,1)^*, g\in C^m(2g,1)^*$, $\mathrm{Tr}_{m+n}([f,g])=0$.

The striking result is the following

THEOREM (2.4.4) (Morita [Mo2] Theorem 6.11): *Each $\mathrm{gr}^m\,\Gamma_{g,1}^*$ is killed by the trace map $\mathrm{Tr}_m$ $(m\geq 3)$.*

By (2.4.2) and (2.4.4), we obtain nontrivial gaps between $\mathrm{gr}^m_{\mathbb{Q}}\,\Gamma_{g,1}^*$ and $\mathrm{gr}^m\,\Gamma_{g,1}^{\mathrm{alg}*}$ for odd $m\geq 3$.

## 3. Lie deformations of Dehn twists

(3.1) The group $\mathrm{gr}^m_{\mathbb{C}}\,\Gamma_{g,n}$ has a natural structure of $\mathrm{Sp}(2g,\mathbb{C})$-module by (2.2.8), hence that of $\mathfrak{sp}(2g,\mathbb{C})$-module, $\mathfrak{sp}(2g,\mathbb{C})$ being the Lie algebra of the

Lie group $\mathrm{Sp}(2g,\mathbb{C})$. Let $\mathfrak{h}$ be the Lie subalgebra consisting of all diagonal matrices in $\mathfrak{sp}(2g,\mathbb{C})$. Let $\varepsilon_i\colon \mathfrak{h} \to \mathbb{C}$ $(1 \le i \le g)$ be the linear map such that

$$\varepsilon_i(H) = h_i \qquad (H \in \mathfrak{h}),$$

$h_i$ being the $(i,i)$-component of $H$.

In this section we assume $g \ge 3$ and shall define elements $\bar{\alpha}_m \in \mathrm{gr}_{\mathbb{Q}}^{1+2m}\Gamma_{g,n}$ and $\bar{\beta}_m \in \mathrm{gr}_{\mathbb{Q}}^{2+2m}\Gamma_{g,n}$ (see Definition (3.10)) to show the following

THEOREM (3.2): *The elements $\bar{\alpha}_m$ and $\bar{\beta}_m$ are non-zero elements of weights $\varepsilon_1^m\varepsilon_2\varepsilon_3$, $\varepsilon_1^m\varepsilon_2^2$ respectively.*

(3.3) We start with certain automorphisms of $\Pi_{g,n}$ corresponding to Dehn twists. Let $\sigma$ and $\tau$ be the unique automorphism of $\Pi_{g,n}$ satisfying

$$\sigma(x_i) = \begin{cases} x_1[x_2, x_{g+2}], & (i = 1), \\ ([x_2, x_{g+2}]^{-1}x_{g+1})x_i([x_2, x_{g+2}]^{-1}x_{g+1})^{-1}, & (i = 2, g+1, g+2), \\ x_i, & (i \ne 1, 2, g+1, g+2); \end{cases}$$

$$\sigma(z_j) = z_j, \qquad (1 \le j \le n);$$

$$\tau(x_i) = \begin{cases} [x_1, x_{g+1}]x_i[x_1, x_{g+1}]^{-1}, & (i = 1, g+1), \\ x_i, & (i \ne 1, g+1); \end{cases}$$

$$\tau(z_j) = z_j, \qquad (1 \le j \le n).$$

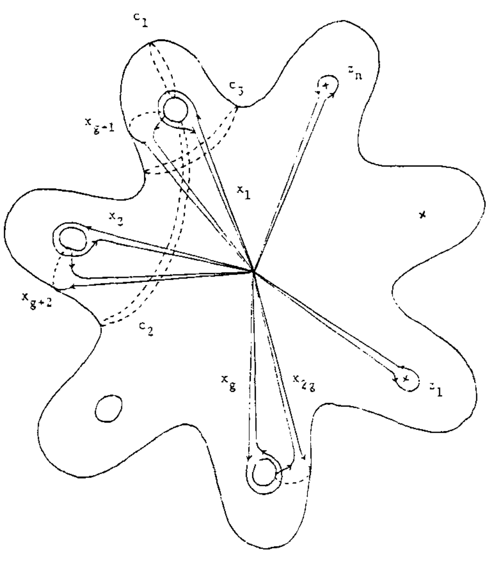


Figure A

The existence and uniqueness of such automorphisms are immediately verified purely algebraically. But we note here the geometric meanings of these automorphisms. The automorphsim $\sigma$ corresponds to (a lifting of) the quotient of two Dehn twists associated with the simple closed curves $c_1$ and $c_2$, and the automorphism $\tau$ corresponds to (a lifting of) the Dehn twist associated with the simple closed curve $c_3$ (Figure A).

From the definition of $\sigma$ and $\tau$, it follows easily that $\sigma \in \tilde{\Gamma}_{g,n}(1)$ and $\tau \in \tilde{\Gamma}_{g,n}(2)$. Recall that an element $\rho$ of $\tilde{\Gamma}_{g,n}(m)$ $(m \geq 1)$ determines naturally a derivation $D_\rho$ of $\operatorname{Gr}\Pi_{g,n}$ of degree $m$ (cf. (2.1.8)). For $\sigma$ and $\tau$, we have

$$
\text{(3.4)} \qquad
\begin{aligned}
D_\sigma(X_i) &= \begin{cases} [X_2, X_{g+2}], & (i = 1), \\ [X_{g+1}, X_2], & (i = 2), \\ [X_{g+1}, X_{g+2}], & (i = g+2), \\ 0, & (i \neq 1, 2, g+2); \end{cases} \\
D_\sigma(Z_j) &= 0, \qquad (1 \leq j \leq n); \\
D_\tau(X_i) &= \begin{cases} [[X_1, X_{g+1}], X_1], & (i = 1), \\ [[X_1, X_{g+1}], X_{g+1}], & (i = g+1), \\ 0, & (i \neq 1, g+1); \end{cases} \\
D_\tau(Z_j) &= 0, \qquad (1 \leq j \leq n).
\end{aligned}
$$

(3.5) Recall that the action of $\tilde{\Gamma}_{g,n}$ on the group $\Pi_{g,n}$ induces a homomorphism

$$\operatorname{Sp}(2g, \mathbb{Z}) \to \operatorname{Aut}(\operatorname{Gr}\Pi_{g,n}).$$

It is easy to see that this induces a Lie algebra homomorphism

$$
\begin{aligned}
\mathfrak{sp}(2g, \mathbb{C}) &\to \operatorname{Der}(\operatorname{Gr}\Pi_{g,n} \otimes \mathbb{C}) \\
L &\mapsto L^*.
\end{aligned}
$$

Similarly, the conjugate action of $\tilde{\Gamma}_{g,n}$ on the Lie algebra $\operatorname{Gr}\tilde{\Gamma}_{g,n}$ induces a Lie algebra homomorphism

$$\mathfrak{sp}(2g, \mathbb{C}) \to \operatorname{Der}(\operatorname{Gr}\Gamma_{g,n} \otimes \mathbb{C}).$$

Recall that the $\operatorname{Sp}(2g, \mathbb{Z})$-module structure of $\operatorname{gr}^m \tilde{\Gamma}_{g,n}$ is described in terms of that of the coordinate module $C^m(2g, n)$. By (2.2.2) and (2.2.5), the action of $\mathfrak{sp}(2g, \mathbb{C})$ on the coordinate $(S_i)_i$ is given by

$$\text{(3.6)} \qquad L.(S_i)_i = (L^* S_i)_i - (S_i)_i L,$$

where $(S_i)_i L$ represents a product as matrices.

(3.7) Let $E_{pq} = (e_{ij})$ denote the element of $\mathfrak{sp}(2g, \mathbb{Q})$ such that

$$e_{ij} = \begin{cases} 1, & (p,q) = (i,j), \\ 0, & \text{otherwise}, \end{cases}$$

and put

$$\begin{aligned} L_1 &= E_{1,g+1}, \\ L_2 &= E_{2,g+3} + E_{3,g+2}, \\ L_3 &= E_{1,g+2} + E_{2,g+1}, \\ L_4 &= E_{g+2,2}. \end{aligned}$$

Let $\bar{\sigma} \in \mathrm{gr}^1 \tilde{\Gamma}_{g,n}$ (resp. $\bar{\tau} \in \mathrm{gr}^2 \tilde{\Gamma}_{g,n}$) be the class of $\sigma$ (resp. $\tau$) defined in (3.3). We define $\alpha \in \mathrm{gr}^1_{\mathbb{Q}} \tilde{\Gamma}_{g,n}$ and $\beta$, $\gamma \in \mathrm{gr}^2_{\mathbb{Q}} \tilde{\Gamma}_{g,n}$ by

$$\alpha = -L_2 L_1 \bar{\sigma}, \quad \beta = \frac{1}{2} L_3^2 \bar{\tau}, \quad \gamma = -\frac{1}{2} L_4 L_3^2 \bar{\tau}.$$

By using (3.4) and (3.6), we have the following

LEMMA (3.8):

(1)

$$D_\alpha(X_i) = \begin{cases} [X_2, X_3] & (i = g+1), \\ -[X_1, X_3] & (i = g+2), \\ [X_1, X_2] & (i = g+3), \\ 0 & (i \neq g+1, g+2, g+3); \end{cases}$$

$$D_\alpha(Z_j) = 0 \qquad (1 \leq j \leq n).$$

(2)

$$D_\beta(X_i) = \begin{cases} [[X_1, X_2], X_2] & (i = g+1), \\ -[[X_1, X_2], X_1] & (i = g+2), \\ 0 & (i \neq g+1, g+2); \end{cases}$$

$$D_\beta(Z_j) = 0 \qquad (1 \leq j \leq n).$$

(3)

$$D_\gamma(X_i) = \begin{cases} [[X_1, X_2], X_1] & (i = 2), \\ [[X_1, X_{g+2}], X_2] + [[X_1, X_2], X_{g+2}] & (i = g+1), \\ [X_1, [X_1, X_{g+2}]] & (i = g+2), \\ 0 & (i \neq 2, g+1, g+2); \end{cases}$$

$$D_\gamma(Z_j) = 0 \qquad (1 \leq j \leq n).$$

COROLLARY (3.9): *The elements $\alpha$, $\beta$, $\gamma$ are eigenvectors of $\mathfrak{h}$ of weights $\varepsilon_1\varepsilon_2\varepsilon_3$, $\varepsilon_1^2\varepsilon_2^2$, $\varepsilon_1^2$, respectively.*

*Definition (3.10):* For each non-negative integer $m$, let

$$\alpha_m := \operatorname{ad}(\gamma)^m(\alpha) \in \operatorname{gr}_{\mathbb{Q}}^{1+2m} \tilde{\Gamma}_{g,n},$$
$$\beta_m := \operatorname{ad}(\gamma)^m(\beta) \in \operatorname{gr}_{\mathbb{Q}}^{2+2m} \tilde{\Gamma}_{g,n},$$

and define $\bar{\alpha}_m$ (resp. $\bar{\beta}_m$) to be the element of $\operatorname{gr}_{\mathbb{Q}}^{1+2m} \Gamma_{g,n}$ (resp. $\operatorname{gr}_{\mathbb{Q}}^{2+2m} \Gamma_{g,n}$) determined by $\alpha_m$ (resp. $\beta_m$). ∎

LEMMA (3.11):

$(1)_{\alpha_m}$ $D_{\alpha_m}(X_1) = D_{\alpha_m}(X_3) = 0.$

$(1)_{\beta_m}$ $D_{\beta_m}(X_1) = D_{\beta_m}(X_3) = 0.$

$(2)_{\alpha_m}$ $D_{\alpha_m}(X_{g+3}) = \operatorname{ad}(X_1)^{2m+1}(X_2).$

$(2)_{\beta_m}$ $D_{\beta_m}(X_{g+2}) = (-2)^m \operatorname{ad}(X_1)^{2m+2}(X_2).$

*Proof:* By the definition of $\alpha_m$ and $\beta_m$, we have

$$D_{\alpha_m} = D_\gamma D_{\alpha_{m-1}} - D_{\alpha_{m-1}} D_\gamma,$$
$$D_{\beta_m} = D_\gamma D_{\beta_{m-1}} - D_{\beta_{m-1}} D_\gamma.$$

From this and Lemma (3.8), all formulas of Lemma (3.11) follow by induction on $m$. ∎

We need the following two lemmas whose proof will be given later.

LEMMA (3.12): *The element* $\operatorname{ad}(X_1)^m(X_2)$ *of* $\operatorname{Gr}\Pi_{g,n}$ *is non-zero for all* $m$.

LEMMA (3.13): *Let $X_p$ and $X_q$ be given ($1 \le p < q \le 2g$). Let $T$ be an element of $\operatorname{Gr}\Pi_{g,n}$ satisfying $[T, X_p] = [T, X_q] = 0$. Then, $T = 0$.*

*(3.14) Proof of Theorem (3.2):* By Corollary (3.9) and the definitions of $\alpha_m$ and $\beta_m$, it follows that $\alpha_m$ and $\beta_m$ are vectors of $\mathfrak{h}$ of weights $\varepsilon_1^{2m+1}\varepsilon_2\varepsilon_3$, $\varepsilon_1^{2m+2}\varepsilon_2^2$ respectively. Hence, it suffices to show that the derivation $D_{\alpha_m}$ and $D_{\beta_m}$ are *not* inner. Assume, on the contrary, that there exists an element $T$ of $\operatorname{Gr}\Pi_{g,n}$ such that $D_{\alpha_m} = \operatorname{ad}(T)$. Then, $T = 0$ by Lemma (3.11) $(1)_{\alpha_m}$ and Lemma (3.13). Hence, $D_{\alpha_m} = 0$. This contradicts Lemma (3.11) $(2)_{\alpha_m}$ and Lemma (3.12). Therefore, $D_{\alpha_m}$ is not inner. Similarly, $D_{\beta_m}$ is not inner and the proof is completed. ∎

*(3.15) Proofs of Lemma (3.12) and Lemma (3.13):* In the case that $n \geq 1$, $\mathrm{Gr}\,\Pi_{g,n}$ is a free Lie algebra generated by $X_1,\ldots,X_{2g},Z_1,\ldots, Z_{n-1}$ over $\mathbb{Z}$ (cf. (2.1.2)). Hence, these two lemmas are direct consequences of the well known fact that

(*) the centralizer of $X_i$ in $\mathrm{Gr}\,\Pi_{g,n}$ coincides with $\mathbb{Z}X_i$.

Thus, to prove these lemmas, we may restrict ourselves to the case of $n = 0$. Since the authors do not know whether (*) holds also in this case, we shall give direct proofs of these lemmas.

*Proof of Lemma (3.12) in the case of $n = 0$:* Let $A$ denote the free associative algebra on $X_1,\ldots, X_{2g}$ over $\mathbb{Z}$ and $\mathfrak{r}$ be the ideal of $A$ generated by

$$R = \sum_{i=1}^{g}(X_iX_{g+i} - X_{g+i}X_i).$$

Then, the universal enveloping algebra of $\mathrm{Gr}\,\Pi_{g,0}$ is canonically isomorphic to the quotient algebra $A/\mathfrak{r}$ ([L]). Thus, it suffices to show that

$$\mathrm{ad}(X_1)^m(X_2) \notin \mathfrak{r}$$

holds in $A$. Assume that this does not hold. Then we have

$$\mathrm{ad}(X_1)^m(X_2) = \sum_r a_rRb_r,$$

where $a_r$ (resp. $b_r$) runs over homogeneous polynomials (resp. monomials) of $A$ satisfying

$$\deg(a_r) + \deg(b_r) = m - 1;$$
$$b_r \neq b_{r'} \quad \text{if } r \neq r';$$
$$\text{the coefficient of } b_r = 1.$$

Easy free differential calculations (cf. (2.4)) on the left-hand side show that

$$\frac{\partial^m}{\partial X_1^m}(\text{L.H.S.}) = (-1)^m X_2.$$

Similary on the right-hand side, we have

$$\frac{\partial^m}{\partial X_1^m}(\text{R.H.S.}) = -a_{r_0}X_{g+1},$$

where $r_0$ is the index such that $b_{r_0} = X_1^{m-1}$. This is a contradiction and the proof is completed. ∎

*Proof of Lemma (3.13) in the case of $n = 0$:* This proof is a refinement of the fact that the Lie algebra $\mathrm{Gr}\,\Pi_{g,0}$ has trivial center. It can be done along the idea of J. Labute (cf. Asada [A1].) Let $S$ be a subset of $\{X_1, \ldots, X_{2g}\}$ satisfying

(1) $\sharp(S \cap \{X_i, X_{g+i}\}) = 1$ for $1 \leq i \leq g$,
(2) $X_p \notin S$,
(3) $X_q \in S$,

and $\mathfrak{a}$ be the ideal of $\mathrm{Gr}\,\Pi_{g,0}$ generated by $S$. Then, the quotient algebra $\mathrm{Gr}\,\Pi_{g,0}/\mathfrak{a}$ is free on (the image of) $\{X_1, \ldots, X_{2g}\} \setminus S$. As

$$[\bar{T}, \bar{X}_p] = 0 \quad \text{in } \mathrm{Gr}\,\Pi_{g,0}/\mathfrak{a}$$

($\bar{\ }$ denotes modulo $\mathfrak{a}$), it follows that $\bar{T} = \lambda \bar{X}_p$ with $\lambda \in \mathbb{Z}$. Since $[T, X_q] = 0$, it follows that $\lambda = 0$, i.e. $T \in \mathfrak{a}$.

On the other hand, by using the elimination theorem of free Lie algebras (cf. [Bo] Chap. 2 §2), we can show that $\mathfrak{a}$ is a free Lie algebra on an infinite set containing $X_q$. As $[T, X_q] = 0$, we have $T = \mu X_q$ with $\mu \in \mathbb{Z}$. Since $[T, X_p] = 0$, it follows that $\mu = 0$, i.e. $T = 0$. ∎

## 4. Proofs of Theorems A and B

*(4.1) Proof of Theorem B:* This is a direct consequence of Theorem (3.2) and Corollary (2.3.7). ∎

(4.2). Using the universal characters explained in (2.3), we have

$$\begin{aligned}
[\mathrm{gr}^1\,\Pi(\pi)] &= [1]_{\mathrm{Sp}} \\
[\mathrm{gr}^2\,\Pi(\pi)] &= [1^2]_{\mathrm{Sp}} + F(\pi)[0]_{\mathrm{Sp}}, \\
[\mathrm{gr}^3\,\Pi(\pi)] &= [2,1]_{\mathrm{Sp}} + F(\pi)[1]_{\mathrm{Sp}}, \\
[\mathrm{gr}^1\,\Gamma(\pi)] &= [1^3]_{\mathrm{Sp}} + F(\pi)[1]_{\mathrm{Sp}}, \\
[\mathrm{gr}^2\,\Gamma(\pi)] &= [2^2]_{\mathrm{Sp}} + F(\pi)[1^2]_{\mathrm{Sp}} + (F(\pi)^2/2 + F(\pi^2)/2 - F(\pi))[0]_{\mathrm{Sp}}, \\
[\mathrm{gr}^3\,\Gamma(\pi)] &= [3,1^2]_{\mathrm{Sp}} + [3]_{\mathrm{Sp}} + F(\pi)[2,1]_{\mathrm{Sp}} + (F(\pi)^2/2 - F(\pi^2)/2)[1]_{\mathrm{Sp}},
\end{aligned}$$

where $[\lambda]_{\mathrm{Sp}}$ is the universal Sp-character corresponding to the partition $\lambda$ introduced in [KT] (see [NT2] for an extended table of these calculations). Recall that, when the length of $\lambda$ is $\leq g$, $[\lambda]_{\mathrm{Sp}}$ is mapped by $\pi_{\mathrm{Sp}(2g)}$ to the irreducible character of $\mathrm{Sp}(2g,\mathbb{C})$ corresponding to the partition $\lambda$ itself. Therefore, it is observed that $[\mathrm{gr}^2_{\mathbb{C}}\,\Gamma^{\mathrm{alg}}_{g,n}(\pi)]$ is stable for $g \geq 2$ and so are $[\mathrm{gr}^1_{\mathbb{C}}\,\Gamma^{\mathrm{alg}}_{g,n}(\pi)]$ and $[\mathrm{gr}^3_{\mathbb{C}}\,\Gamma^{\mathrm{alg}}_{g,n}(\pi)]$ for $g \geq 3$. Estimating the $S_n$-representations occurring in multiplicities of each irreducible Sp-representations, we conclude the following $\mathrm{Sp}(2g,\mathbb{C}) \times S_n$-isomorphisms:

$$\begin{aligned}
\mathrm{gr}^1_{\mathbb{C}}\,\Gamma^{\mathrm{alg}}_{g,n} &\cong [1^3]_{\mathrm{Sp}(2g)} + \{(n) + (n-1,1)\} \otimes [1]_{\mathrm{Sp}(2g)} \quad (g \geq 3),\\
\mathrm{gr}^2_{\mathbb{C}}\,\Gamma^{\mathrm{alg}}_{g,n} &\cong [2^2]_{\mathrm{Sp}(2g)} + \{(n) + (n-1,1)\} \otimes [1^2]_{\mathrm{Sp}(2g)}\\
&\qquad + \{(n) + (n-1,1) + (n-2,2)\} \otimes [0]_{\mathrm{Sp}(2g)} \quad (g \geq 2),\\
\mathrm{gr}^3_{\mathbb{C}}\,\Gamma^{\mathrm{alg}}_{g,n} &\cong [3,1^2]_{\mathrm{Sp}(2g)} + [3]_{\mathrm{Sp}(2g)} + \{(n) + (n-1,1)\} \otimes [2,1]_{\mathrm{Sp}(2g)}\\
&\qquad + \{(n-2,1^2) + (n-1,1)\} \otimes [1]_{\mathrm{Sp}(2g)} \quad (g \geq 3).
\end{aligned}$$

Here we denote by $(\mu)$ the irreducible $S_n$-representation corresponding to $\mu$ subject to Murnaghan's rule for disordered partitions (cf. [Mu], p.461). Since every irreducible component has multiplicity one, these $\mathrm{Sp}(2g,\mathbb{C}) \times S_n$-isomorphisms are defined over $\mathbb{Q}$. By virtue of Lemma (2.2.8), it remains for the proof of Theorem A to determine which irreducible component of $\mathrm{gr}^m\,\Gamma^{\mathrm{alg}}_{g,n}$ is contained in the image of $\mathrm{gr}^m_{\mathbb{Q}}\,\Gamma_{g,n}$ and which is not for $m = 1, 2, 3$.

*(4.3) Proof of Theorem A:* We first consider the case $n = 0, 1$. By a result of Johnson ([J]), we have $\mathrm{gr}^1_{\mathbb{Q}}\,\Gamma_{g,n} \cong \mathrm{gr}^1\,\Gamma^{\mathrm{alg}*}_{g,n}$ $(n = 0, 1)$, and by a result of Morita ([Mo1] Proposition 1.2) we see that $\mathrm{gr}^2_{\mathbb{Q}}\,\Gamma^*_{g,1} \cong \mathrm{gr}^2\,\Gamma^{\mathrm{alg}*}_{g,1}$. Moreover, the forgetful homomorphism induces a sequence of $\mathrm{Sp}(2g,\mathbb{Q})$-modules

$$0 \to \mathrm{gr}^m_{\mathbb{Q}}\,\Pi_{g,0} \to \mathrm{gr}^m_{\mathbb{Q}}\,\Gamma_{g,1} \to \mathrm{gr}^m_{\mathbb{Q}}\,\Gamma_{g,0} \to 0 \tag{4.3.1}$$

which is exact at least for $1 \leq m \leq 3$ (cf. [A2]). Our result for $\mathrm{gr}^m_{\mathbb{Q}}\,\Gamma_{g,n}$ $(n = 0, 1$, $m = 1, 2)$ immediately follows from these facts together with (2.2.8). For $m = 3$, by Morita's results (2.4.2),(2.4.4), $\mathrm{Tr}_3 \colon \mathrm{gr}^3\,\Gamma^{\mathrm{alg}*}_{g,1} \to \mathrm{Sym}^3 H \otimes \mathbb{Q} (\cong [3]_{\mathrm{Sp}(2g)})$ is surjective and $\mathrm{gr}^3_{\mathbb{Q}}\,\Gamma^*_{g,1}$ is contained in this kernel. By Theorem B, there is a nonzero component of type $[3,1^2]_{\mathrm{Sp}}$ in $\mathrm{gr}^3_{\mathbb{Q}}\,\Gamma_{g,n}$. Thus, $\mathrm{gr}^3_{\mathbb{Q}}\,\Gamma_{g,n}$ $(n = 0, 1)$ have expected components.

Next we consider general cases. If $n \geq 1$ and $m = 1, 2, 3$, we have the following commutative diagrams of exact sequences of $\mathrm{Sp}(2g,\mathbb{Q}) \times S_n$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{gr}^m_{\mathbb{Q}}\,\Pi_{g,n} & \longrightarrow & \mathrm{gr}^m_{\mathbb{Q}}\,\Gamma_{g,n+1} & \longrightarrow & \mathrm{gr}^m_{\mathbb{Q}}\,\Gamma_{g,n} & \longrightarrow & 0\\
 & & \downarrow \wr & & \downarrow \iota_m & & \downarrow \iota_m & & \\
0 & \longrightarrow & \mathrm{gr}^m_{\mathbb{Q}}\,\Pi_{g,n} & \longrightarrow & \mathrm{gr}^m\,\Gamma^{\mathrm{alg}}_{g,n+1} & \longrightarrow & \mathrm{gr}^m\,\Gamma^{\mathrm{alg}}_{g,n} & \longrightarrow & 0
\end{array}$$

(cf. [A2]). From this it follows inductively that any gap between $\mathrm{gr}^m_{\mathbb{Q}}\Gamma_{g,n}$ and $\mathrm{gr}^m\Gamma^{\mathrm{alg}}_{g,n}$ must come from what appears in the case of $n=1$. Thus the argument is reduced to the above case, and the proof of Theorem A is completed. ∎

*Remark (4.4):* We complement remarks on Theorems A, B for the case $g=2$. In Theorem A, we have only to drop the components $[1^3]$, $[3,1^2]$ from the right hand sides of (1),(3) respectively. In Theorem B, the statement (2) holds true provided that $n\geq 1$, for the construction of $\bar{\beta}$, $\bar{\gamma}$ in §3 together with Corollary (2.3.7) for even $m$ are also valid in the case $g=2$, $n\geq 1$. ∎

*Remark (4.5):* The authors do not know whether the sequence (4.3.1) is exact for $m\geq 4$. We remark that this is closely related to how large the image of the homomorphism $\iota_m$ is. (See (2.2.7).) Here we only mention that *if* (4.3.1) *is exact for* $m=4$, *then the image of* $\mathrm{gr}^4_{\mathbb{Q}}\Gamma_{g,1}$ *by* $\iota_4\otimes\mathbb{Q}$ *cannot fill* $\mathrm{gr}^4\Gamma^{\mathrm{alg}}_{g,1}$. (Note that Theorem (2.4.4) gives no restriction on that image.) In fact, if the image of $\iota_4\otimes\mathbb{Q}$ coincides with $\mathrm{gr}^4\Gamma^{\mathrm{alg}}_{g,1}$, by the exactness of (4.3.1) for $m=4$, we have

$$\dim \mathrm{gr}^4_{\mathbb{Q}}\Pi_{g,0}=\dim \mathrm{gr}^4\Gamma^{\mathrm{alg}}_{g,1}-\dim \mathrm{gr}^4_{\mathbb{Q}}\Gamma_{g,0}\geqq \dim \mathrm{gr}^4\Gamma^{\mathrm{alg}}_{g,1}-\dim \mathrm{gr}^4\Gamma^{\mathrm{alg}}_{g,0}.$$

By easy calculations using a formula of Labute (2.1.3), we see that the right hand side is equal to $\dim \mathrm{gr}^4_{\mathbb{Q}}\Pi_{g,0}+2g^2+g$ (cf. also [NT2]). This is a contradiction. ∎

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